Resolution of the Ehrenfest Paradox by New Contraction (Expansion) Criteria

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If we assume that the integrity of an elastic solid is governed by exchanges of real and virtual photons, the solid becomes a seat of innumerable interconnected microscopic Kennedy-Thorndike experiments. The null result of that experiment leads to a deformation criterion having the Lorentz-Fitzgerald contraction as a special case. For a spinning disc the criterion, combined with an equation of compatibility, gives radial and tangential deformations which resolve the Ehrenfest paradox.

I. Introduction

If a Lorentz-Fitzgerald contraction exists, the circumference of a rotating disc, being parallel to the velocity, will shrink, whereas a radius, being perpendicular to the velocity, will not. This is the Ehrenfest paradox.

Ives [1] avoided the paradox by postulating that a rotating disc becomes dish-shaped and obtained an equation defining the new shape. An objection to this idea is that it violates the symmetry of the situation, since there is no a priori reason why the disc should "dish" one way or the other. Winterberg [2] has called my attention to the possible application of spontaneous symmetry breaking to resolve this ambiguity. For a thin disc irregularities would determine the direction of the dishing. The concept is unsuitable for a thick disc. Eddington [3], by methods of general relativity, concluded that elements of the disc contract both radially and peripherally by the factor $(1 - \frac{1}{4} v^2/c^2)^{1/2}$. Since it brings out a point which is central to the following discussion, let us examine Eddington's result. The circumference of a spinning disc would be

$$C = 2\pi R = 2\pi \int_0^{R_0} [1 - (1/4)(\omega^2 r^2/c^2)]^{1/2} \, dr,$$

where R_0 is the radius of the disc at rest and ω the angular speed. Expanding the radical, integrating and retaining only two terms, we get

$$C = 2\pi R_0 - (\pi/12)(\omega^2/c^2) R_0^3 + \cdots.$$

On the other hand, elements in the peripheral direction have contracted to give

$$C = 2\pi R_0 [1 - \frac{1}{4}(\omega^2/c^2) R_0^2]^{1/2}$$
$$= 2\pi R_0 - (\pi/4)(\omega^2/c^2) R_0^3 + \cdots.$$

The two C's do not agree, so that the paradox, which is purely geometrical and has nothing to do with material properties, persists in a varied form in Eddington's analysis.

What seems to be required are a deformation criterion and a compatibility relation which will yield the same value of C regardless of which way it is computed.

The purpose of this paper is to describe one way of doing this. The initial idea is that the integrity of an elastic solid, deriving as it does from electromagnetic forces having the photon as the exchange particle, depends upon, or is controlled by, a highly organized exchange of radiation (real and virtual photons). The basic structural component of a solid can be regarded as a triangle formed by any three non-collinear atoms so that for simplicity we may first consider the behavior of photons travelling back and forth between atoms at points 0, A, and B, where 0 is the origin, A is on the y-axis and B on the x-axis. We confine our attention to radiation making the round trips 0A0 and 0B0. It is reasonable to assume that for the "highly organized" radiation exchange, the phase differences at 0 between the 0A0 and 0B0 photons are important; for structural integrity the differences should be independent of the motion of the body.

That is, if the waves are in phase when the body is at rest, they are in phase after it has been set into motion. A more definite description of the situation is that a large number of Kennedy-Thorndike experi-

ments, each on the scale of a few interatomic distances, is being performed throughout the body.

A simple review of the Kennedy-Thorndike experiment is as follows: With the object at rest, a wave train leaves 0 along A and returns to 0. Another wave train leaves 0 along B and returns to 0. Suppose that upon their return to 0 the two trains are in phase. If 0A is greater than 0B, the A wave-crest which ultimately meets the B wavecrest at 0 must have left 0 at an earlier time. This time difference corresponds to n_0 wavelengths. As the body changes velocity continuously from rest, n_0 cannot change to a new value n, for if it did the interference fringes would shift, contrary to the null result of the Kennedy-Thorndike experiment. In terms of structure we might interpret a change from n_0 to a different n as a great number of microscopic catastrophes which threaten or destroy the integrity of the body. That is, we do not assume a Lorentz-Fitzgerald contraction (or any other dimension change) ab initio. Rather, the basic assumption is the permanence of n_0.

II. The Kennedy-Thorndike Experiment

In triangle 0AB, let 0A and 0B be of rest lengths L_0 and l_0, respectively. The transit times for light along paths 0A0 and 0B0 are $t_A = 2L_0/c$ and $t_B = 2l_0/c$, respectively. The value of n_0 is $v_0(t_A - t_B) = (2v_0/c)(L_0 - l_0)$.

Suppose that when the triangle is moving with speed u along 0B the lengths change to

$$L = gL_0; \quad l = f l_0.$$

The round-trip transit times then become

$$t'_A = 2gL_0\gamma/c; \quad t'_B = 2fl_0\gamma^2/c,$$

where $\gamma = [1 - (u/c)^2]^{-1/2}$. Then

$$t'_A - t'_B = (2\gamma/c)(gL_0 - \gamma f l_0).$$

For the wave trains to return to 0 in phase, the A wave crest must have left 0 before the B wave crest by n wavelengths, where

$$n = v(t'_A - t'_B) = (2\gamma v/c)(gL_0 - \gamma f l_0).$$

Here we have allowed for a possible change in the frequency from v_0 to v. It is well-known that to interpret the Kennedy-Thorndike experiment through deformations, both a deformation and frequency change are needed. The frequency change is sometimes described as an "ad hoc hypothesis" [4]. However, the

pejorative ad hoc has been shown to be incorrect by the demonstration [5] that the frequency change (time dilation) occurs as a cumulative result of reflections at moving mirrors during the acceleration of the interferometer, the effect involving the Lorentz contraction and the tilting of mirrors necessary to prevent the wave-train (photon) from escaping. Since $n = n_0$ by hypothesis, we get

$$v = v_0(L_0 - l_0)/\gamma(gL_0 - \gamma f l_0). \tag{1}$$

Since this is true in general it is true in particular for a Michelson-Morley experiment where $L_0 = l_0$. The numerator vanishes and the denominator, since v does not vanish, must do likewise, so that

$$g = \gamma f. \tag{2}$$

The Lorentz-Fitzgerald contraction corresponds to $g = 1$ and $f = 1/\gamma$. From (1) we get the time dilation effect, $v = v_0/\gamma^2 f = v_0/\gamma$.

The choice $g = 1$ for rectilinear motion can be defended as follows: it is shown in reference [5] that a photon trapped in a transversely-moving light clock of constant length changes frequency during acceleration in accordance with the relativistic Doppler formula. Thus its energy increase is equal to the work done on it during acceleration. Now, if the connecting rod changes length (thereby giving the mirrors normal velocities) work is done on or by the photon at reflections, so that it no longer has the correct energy value.

III. The Compatibility Equation

The simple choice $g = 1$ is not available for rotational motion if we are to avoid the paradox, so that we look for a new criterion that is consistent with (2).

Let us start with a disc with a rest radius x. The disc is assumed to be rigid except for the effects to be discussed. Stated otherwise, strain from inertial forces is neglected. When spinning about the axis with angular velocity ω each area element will change its dimensions, each radial length dr becoming $g\,dr$. The new radius will then be $\int_0^x g\,dr$. The new circumference calculated from the peripheral changes is $2\pi x f$ and from the radial changes is $2\pi \int_0^x g\,dr$. For these to have the same value we must have, using (2):

$$x f(x) = \int_0^x g(r)\,dr = \int_0^x \gamma(r)\,f(r)\,dr, \tag{3}$$

where
$$\gamma(r) = [1 - (\omega^2 r^2/c^2)]^{-1/2}.$$

Since x can assume any value, provided $\omega x < c$, we may differentiate (3) to get

$$x f'(x) + f(x) = \gamma(x) f(x).$$

Rearranging and integrating;

$$df/f = (1/x)(\gamma - 1)\, dx,$$
$$\ln x f = \int (\gamma/x)\, dx + \ln K.$$

The solution is readily found to be

$$f = (K\omega/c)/[1 + (1 - \omega^2 x^2/c^2)^{1/2}], \qquad (4)$$

where K is a constant of integration.

For small values of x the speed is low so that as $x \to 0$, $f \to 1$ and (4) becomes $1 = K\omega/2c$. Thus, the final form of (4) is

$$f = 2/(1 + \gamma^{-1}) = 2\gamma/(\gamma + 1), \qquad (5)$$

while

$$g = \gamma f \quad = 2\gamma^2/(\gamma + 1),$$
$$v = v_0/\gamma^2 f = v_0(\gamma + 1)/2\gamma^3. \qquad (6)$$

Since γ is always greater than unity, both f and g are also. That is, there are both radial and circumferential expansions, but no geometrical paradox.

The above treatment contains the implicit hypothesis that a Kennedy-Thorndike experiment yields a null result for rotational motion. This is plausible for small interferometers (arms having lengths of a few interatomic distances) and is supported by the experimental evidence that the clock hypothesis (that clock rate is affected by acceleration only through its affect on velocity) is correct.

IV. Discussion

Objections to the Lorentz-Fitzgerald contraction hypothesis as a real physical effect, measurable in principle within a single reference frame, on the grounds that it is "contrived" or ad hoc, ignore a salient point, namely, that in denying the contraction one is making the implicit hypothesis of a zero contraction. Since there is not a shred of evidence to support this, it is as much open to the ad hoc label as any other deformation.

Indeed, prior to 1892, it is doubtful if the length of an object in motion relative to the laboratory was measured at all, much less with any precision, so that the assumption that it remained constant, although accepted by everyone, was purely gratuitous. For all they knew, a train travelling at sixty miles per hour might have changed length by a meter or two! Nevertheless, the Ehrenfest paradox furnishes a cogent objection to the Lorentz-Fitzgerald contraction as the *only* deformation criterion.

The treatment outlined above, while reducing to the Lorentz-Fitzgerald contraction as an important special case, avoids the paradox. The underlying concept is connectivity supplied by an interlaced set of many "interferometers" each created by three atoms of a solid. The notion of connectivity in relation to special relativity has previously been used in an interesting paper by Dewan and Beran [6]. They discuss the difference between a meter stick which has been set into motion (contraction) and two identical rockets a meter apart and fired simultaneously (no contraction of the separation distance). They attribute the difference to the fact that there is connectivity in the case of the meter stick but none in the case of the rockets.

I have tried to make the connectivity concept more specific by introducing to-and-fro flights of photons as the connecting mechanism, thereby enabling us to invoke the unchallenged null result of the Kennedy-Thorndike experiment.

Since, as shown in reference 5, the frequency changes of the photons occur as a consequence of reflections at moving mirrors, the nature of connectivity from this viewpoint is seen to be essentially dynamic.

As a sort of bonus, the idea that connectivity results from photon exchanges supplies a reason why c is such a special velocity. An inertial frame is not a set of isolated clocks at rest relative to one another in otherwise empty space, but rather a set connected to one another by a material framework. Thus c is an inherent property of the inertial frame itself. Even if tachyons were discovered, or if clock synchronization were to be effected without using light signals (e.g., slow transport of clocks) c would still retain its unique position in the Lorentz transformations and wherever properties of matter are involved.

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